The β^+ Decay of a $0^+ \longrightarrow 0^+$ Transition

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Abstract

The derivation of the differential decay rate for a $0^+ \longrightarrow 0^+$ decay is given. Recoil order corrections are included and, where possible, so are massive neutrinos. Familiarity with the Dirac equation and Feynman diagrams is assumed; however, given that, the steps in the calculations are explicitly written out in gruesome detail so that the reader can easily follow the derivation.

The main purpose of this report is to formally outline the theory behind the β^+ decay of a superallowed decay (*i.e.* $^{38\text{m}}$K). The other purpose of this paper is to clarify what should be — and help explain what has been — put into my Monte Carlo for TRINAT's $\beta - \nu$ correlation experiment.

Developments towards including polarization have been described in an earlier paper [1]. If I ever find the time, I'll merge them and embellish the theory for GT decays; so then I would describe all of TRINAT's $^{36,\ 37,\ 38\text{m}}$K experiments and explain how it they are all implemented under my MC.

Contents

1 Notation and Convention

As per Halzen and Martin [2], we take the γ-matrices to be:

$$\gamma_0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{1}$$

where $\vec{\sigma}$ are the Pauli spin matrices and $\mathbb{I}$ is the unit 2×2 matrix.

We will also make use of the Dirac spinors

$$u_r(p) = \sqrt{E+m} \begin{pmatrix} \chi_r \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_r \end{pmatrix}, \qquad E > 0, r = 1, 2 \tag{2a}$$

$$v_r(p) = \sqrt{|E|+m} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{|E|+m}\phi_r \\ \phi_r \end{pmatrix}, \qquad E < 0, r = 1, 2 \tag{2b}$$

where the χ and ϕ spinors carry the spin of particles and anti-particles respectively; explicitly, they are given by:

$$\overbrace{\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\text{spin up}}, \quad \underbrace{\chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{spin down}}, \quad \text{and} \quad \underbrace{\phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{spin down}}, \quad \overbrace{\phi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{\text{spin up}}. \tag{3}$$

The Dirac wavefunctions are normalized such that:

$$u^\dagger u = v^\dagger v = 2E, \tag{4a}$$

$$\overline{u}u = -\overline{v}v = 2m \tag{4b}$$

The adjoints for wavefunctions (ψ) and operators (Γ) are:

$$\overline{\psi} = \gamma^0 \psi^\dagger \tag{5a}$$

$$\text{and} \quad \overline{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0 \tag{5b}$$

Some useful relations include:

$$(\gamma^0)^2 = -\boldsymbol{\gamma}^2 = \gamma_5^2 = \mathbb{I} \tag{6a}$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0, \qquad (\gamma_5)^\dagger = \gamma_5 \tag{6b}$$

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu, \qquad (\overline{u}_r(p))^\dagger = \gamma^0 u_r(p) \tag{6c}$$

$$\sum_{\text{spins}} u_r(p)\overline{u}_r(p) = \not{p} + m, \tag{6d}$$

$$\text{and} \qquad \sum_{\text{spins}} v_r(p)\overline{v}_r(p) = -\not{p} + m, \tag{6e}$$

$$\text{where} \qquad \not{p} \equiv p_\mu\gamma^\mu. \tag{6f}$$

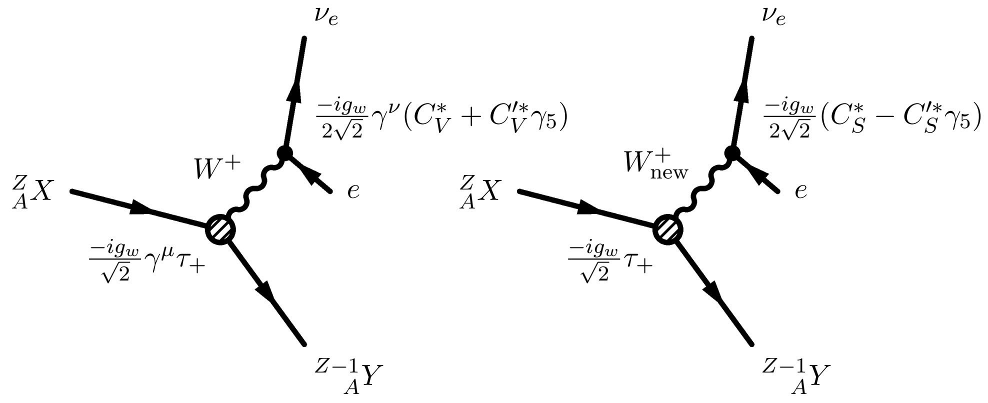
Throughout this report, terms of order $1/M$ and $1/M'$ are retained while higher powers (e.g. E_e^2/M^2 and k^2/M'^2) are considered negligible; any time an approximation is applied, it will be noted by $\approx$ if not stated explicitly. All approximations are made to first order in their Taylor expansions; the three used in this report are:

$$(1 \pm x)^{-\frac{1}{2}} = 1 \pm \tfrac{1}{2}x + \mathcal{O}(x^2) \tag{7a}$$

$$(1 \pm x)^{\frac{1}{2}} = 1 \mp \tfrac{1}{2}x + \mathcal{O}(x^2) \tag{7b}$$

$$\text{and} \quad (1 \pm x)^{-1} = 1 \mp x + \mathcal{O}(x^2) \tag{7c}$$

Finally, I work in units where $\hbar = c = 1$; but I do *not* set $m_e = 1$.



(a) Vector: $\frac{-i\left(g_{\mu\nu}-\frac{Q_\mu Q_\nu}{M_W^2}\right)}{Q^2-M_W^2}$ (b) Scalar: $\frac{-i}{Q^2-M_W^2}$

Figure 1: General Feynman diagrams for the $0^+ \longrightarrow 0^+$ β^+ decay of $^Z_AX \longrightarrow {}^{Z-1}_{\ \ A}Y + e^+ + \nu_e$ as mediated by a massive boson. The expressions for the propagators for both the vector and scalar interactions are given below their diagrams. For β^- decay, we need to lower isospin in the hadron current ($\tau_+ \to \tau_-$), change the direction of the lepton current and take the Hermition conjugate of its vertex parameters (giving $\frac{-ig_w}{2\sqrt{2}}\gamma^\nu(C_V + C_V'\gamma_5)$ and $\frac{-ig_w}{2\sqrt{2}}\gamma^\nu(C_S + C_S'\gamma_5)$).

2 The Matrix Element of the Decay

The matrix element for β^+ decay is:

$$\mathcal{M}_{\rm fi} = \frac{1}{\sqrt{2}}\big(\mathcal{M}_V + \mathcal{M}_S\big). \tag{8}$$

where $\mathcal{M}_V$ and $\mathcal{M}_S$ are calculated using the Feynman diagrams of Figure 1. The four-momenta of the particles are:

$$\begin{aligned} p &= (M, \vec{0}), && \text{decay is from rest} \\ k &= (E', -\vec{k}), && |\vec{k}| = \sqrt{E'^2 - M'^2} \\ p_e &= (E_e, -\vec{p}_e), && |\vec{p}_e| = \sqrt{E_e^2 - m_e^2} \\ \text{and} \quad p_\nu &= (E_\nu, -\vec{p}_\nu), && |\vec{p}_\nu| = \sqrt{E_\nu^2 - m_\nu^2}. \end{aligned}$$

First, we use the diagram depicted in Figure 1(a) to get the matrix element for vector interactions:

$$\begin{aligned} -i\mathcal{M}_V = \overline{\psi}_Y(k)\big[\tfrac{-ig_w}{\sqrt{2}}\gamma^\mu\tau_+\big]\psi_X(p) \ \times \frac{-i\big(g_{\mu\nu} - \frac{Q_\mu Q_\nu}{M_W^2}\big)}{Q^2 - M_W^2} \\ \times\, \overline{\psi}_e(p_e)\big[\tfrac{-ig_w}{2\sqrt{2}}\gamma^\nu(C_V^* + C_V'^*\gamma_5)\big]\psi_\nu(p_\nu). \end{aligned}$$

It is safe to assume that the momentum transfer, $Q^2 \approx (5\text{ MeV})^2 \ll (80\text{ GeV})^2 \approx M_W^2$. In this limit, the propagators reduce to constants: $\frac{ig_{\mu\nu}}{M_W^2}$ and $\frac{-i}{M_W^2}$. Therefore, we get for vector interactions

$$\mathcal{M}_V \xrightarrow[Q^2 \ll M_W^2]{} \overline{\psi}_Y(k)\gamma^\mu\tau_+\psi_X(p)\ \frac{-g_w^2 g_{\mu\nu}}{4M_W^2}\ \overline{\psi}_e(p_e)\gamma^\nu\big(C_V^* + C_V'^*\gamma_5\big)\psi_\nu(p_\nu) \tag{9}$$

For scalar interactions, Figure 1(b) gives:

$$\begin{aligned} -i\mathcal{M}_S = \overline{\psi}_Y(k)\big[\tfrac{-ig_w}{\sqrt{2}}\tau_+\big]\psi_X(p)\ \frac{-i}{Q^2 - M_W^2} \\ \times\, \overline{\psi}_e(p_e)\big[\tfrac{-ig_w}{2\sqrt{2}}(C_S^* - C_S'^*\gamma_5)\big]\psi_\nu(p_\nu) \end{aligned}$$

$$\mathcal{M}_S \xrightarrow[Q^2 \ll M_W^2]{} \overline{\psi}_Y(k)\tau_+\psi_X(p)\ \frac{-g_w^2}{4M_W^2}\ \overline{\psi}_e(p_e)\big(C_S^* - C_S'^*\gamma_5\big)\psi_\nu(p_\nu) \tag{10}$$

Here we have assumed that the scalar and vector couplings are equal (g_w) and that the mass of the new boson $W^+_{\rm new} = W^+$, the Standard Model vector boson.

2.1 The Hadron Current

Now let us concentrate on the part involving the hadrons. Let us for simplicity assume that the nucleus can be described as a Dirac particle, with minor adjustments due to nuclear structure and isospin selection rules. With that in mind, we break up the nuclear wavefunctions as:

$$\begin{aligned} \psi_X(p) &= u_s(p)\ \psi_{\text{spa}}\psi_{\text{iso}} \\ &= \sqrt{2M}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{2M}\chi_s \end{pmatrix}\ \psi_{\text{spa}}\psi_{\text{iso}} \end{aligned} \tag{11}$$

$$\begin{aligned} \text{and}\quad \overline{\psi}_Y(k) &= \overline{\psi'}_{\text{spa}}\overline{\psi'}_{\text{iso}}\overline{u}_r(k) \\ &= \overline{\psi'}_{\text{spa}}\overline{\psi'}_{\text{iso}}\sqrt{E'+M'}\begin{pmatrix} \chi_r^\dagger & \frac{-\vec{\sigma}\cdot\vec{k}}{E'+M'}\chi_r^\dagger \end{pmatrix} \end{aligned} \tag{12}$$

Here, $\psi_{\text{spa}}, \psi'_{\text{spa}}$ represent our lack of knowledge of the spatial structure of the nuclear states, and $\psi_{\text{iso}}, \psi'_{\text{iso}}$ are the isospins of the system. To deal with these nuclear effects, and since Gamow-Teller decays (which flip the spin of the decaying nucleon) are a different class from the Fermi ones, one defines the Fermi matrix element:

$$M_F = \langle\psi'_{\text{iso}}|\tau_\pm|\psi_{\text{iso}}\rangle\ \langle\psi'_{\text{spa}}|\psi_{\text{spa}}\rangle. \tag{13}$$

If my guess is right, the Gamow-Teller one would be given by:

$$M_{GT} = \langle\psi'_{\text{iso}}|\tau_\pm|\psi_{\text{iso}}\rangle\ \langle\psi'_{\text{spa}}|\vec{\sigma}|\psi_{\text{spa}}\rangle, \tag{14}$$

but this is only a guess!!

The strong force is invariant under rotations in isospin space, so it doesn't care that a proton has transformed into a neutron; since the spin and parity of these are both 0^+, we don't expect that the wavefunction of the decaying nucleon (or the rest of the nucleus) will be signficantly perturbed during the transition. Therefore, this overlap integral should be very close to what we assume will be unity. In order to calculate M_F, we just need the isospin component, which is easily calculable in the $0^+ \longrightarrow 0^+$ cases ($T = 1, T_3 = -1 \longrightarrow 0$):

$$M_F = \left\langle 1\ \ 0\ \middle|\sqrt{T(T+1) - T_3(T_3 \pm 1)}\middle|\ 1 - 1\right\rangle \tag{15}$$

$$= \sqrt{1(2) + 1(0)}\ \left\langle 1\ 0\ |\ 1\ 0\right\rangle \tag{16}$$

$$M_F = \sqrt{2} \tag{17}$$

OK. We now have expressions for the nuclear wavefunctions, and so can evaluate the hadron currents, $\mathcal{M}_V$ and $\mathcal{M}_S$. We get the hadron component of the matrix element of the decay by substituting Eqs. (9) and (10) into Eq. (8); but before we evaluate this: let it be noted that for true generality, one should take the hadron current to be:

$$\begin{aligned} \overline{u}_r(k)\Big[&F_1\gamma^\mu + \frac{F_2}{M}\sigma^{\mu\nu}Q_\nu + i\frac{F_3}{M}Q^\mu \\ &+ G_1\gamma^\mu\gamma^5 + \frac{G_2}{M}\sigma^{\mu\nu}Q_\nu\gamma^5 + i\frac{G_3}{M}Q^\mu\gamma^5\Big]u_s(p), \end{aligned} \tag{18}$$

but this is a *lot* more complicated and best left to the theorists ... see the paper by Nieto [3] (which doesn't even consider scalar components!).

So, armed with simplifications (11) and (12), we begin evaluating the matrix elements:

$$\begin{aligned} &\underline{\text{Scalar}}: \\ &\mathcal{M}_S^{\text{had}} = \langle\psi'_{\text{iso}}|\tau_+|\psi_{\text{iso}}\rangle\ \langle\psi'_{\text{spa}}|\psi_{\text{spa}}\rangle\ \times\ \overline{u}_r(k)u_s(p) \\ &\quad = M_F\ \times\ \sqrt{2M}\sqrt{E'+M'}\begin{pmatrix} \chi_r^\dagger & \frac{-\vec{\sigma}\cdot\vec{k}}{E'+M'}\chi_r^\dagger \end{pmatrix}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{2M}\chi_s \end{pmatrix} \\ &\quad = M_F\sqrt{2M}\sqrt{E'+M'}\Big[\Big(1 - \frac{(\vec{\sigma}\cdot\vec{k})(\vec{\sigma}\cdot\vec{p})}{2M(E'+M')}\Big)\chi_r^\dagger\chi_s\Big]. \end{aligned} \tag{19}$$

The product of the initial and final spins is 1 if $r = s$, otherwise it is zero. Thus scalar hadron currents require the spin does not change in the decay.

$$\underline{\text{Vector}}: \quad \text{case (a)} \qquad \mu = 0$$

$$\mathcal{M}_V^{\text{had}} = M_F\sqrt{2M}\sqrt{E'+M'}\begin{pmatrix}\chi_r^\dagger & \frac{-\vec{\sigma}\cdot\vec{k}}{E'+M'}\chi_r^\dagger\end{pmatrix}\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{2M}\chi_s\end{pmatrix}$$

$$= M_F\sqrt{2M}\sqrt{E'+M'}\Big[\Big(1+\frac{(\vec{\sigma}\cdot\vec{k})(\vec{\sigma}\cdot\vec{p})}{2M(E'+M')}\Big)\chi_r^\dagger\chi_s\Big]. \tag{20}$$

Here, as with the scalar case, the spin is not changed in the decay.

$$\underline{\text{Vector}}: \quad \text{case (b)} \qquad \mu = i$$

$$\mathcal{M}_V^{\text{had}} = M_F\sqrt{2M}\sqrt{E'+M'}\begin{pmatrix}\chi_r^\dagger & \frac{-\vec{\sigma}\cdot\vec{k}}{E'+M'}\chi_r^\dagger\end{pmatrix}\begin{pmatrix}0 & \vec{\sigma}\\ -\vec{\sigma} & 0\end{pmatrix}\begin{pmatrix}\chi_s\\ \frac{\vec{\sigma}\cdot\vec{p}}{2M}\chi_s\end{pmatrix}$$

$$= M_F\sqrt{2M}\sqrt{E'+M'}\Big[\Big(\frac{\vec{\sigma}\cdot\vec{p}}{2M}+\frac{\vec{\sigma}\cdot\vec{k}}{E'+M'}\Big)\chi_r^\dagger\vec{\sigma}\chi_s\Big]. \tag{21}$$

This time, the Pauli spin matrices *do* change the spin, *i.e.* $\chi_r^\dagger\vec{\sigma}\chi_s \neq 0$ only if $r \neq s$. As we are considering $0^+ \longrightarrow 0^+$ decays, this term will be zero and we can ignore it. We therefore only have to consider the time component of the Hamiltonian Eq. (8).

In the limit that $|\vec{k}| \ll 2M$ and $|\vec{p}| \ll E'+M'$, the hadron currents for both scalar and vector transitions are the same:

$$\mathcal{M}_S^{\text{had}} \text{ and } \mathcal{M}_V^{\text{had}} \xrightarrow[|\vec{k}|,|\vec{p}|\ll M,M']{} M_F\sqrt{2M}\sqrt{E'+M'}. \tag{22}$$

2.2 Trace Analysis

Going back to our invariant amplitudes and including the leptons:

$$\mathcal{M}_S = M_F\sqrt{2M(E'+M')}\frac{g_w^2}{4M_W^2}\overline{u}_r(p_\nu)\big(C_S^* - C_S'^*\gamma_5\big)v_s(p_e), \tag{23}$$

$$\mathcal{M}_V = M_F\sqrt{2M(E'+M')}\frac{g_w^2 g_{0\nu}}{4M_W^2}\overline{u}_r(p_\nu)\gamma^\nu\big(C_V^* + C_V'^*\gamma_5\big)v_s(p_e). \tag{24}$$

Also, let us put the strength of the weak interaction in terms of Fermi's coupling constant:

$$\frac{g_w^2}{8M_W^2} = \frac{G_F}{\sqrt{2}} \quad \Longrightarrow \quad \Big(\frac{g_w^2}{4M_W^2}\Big)^2 = 2G_F^2 \tag{25}$$

Upon squaring the matrix element, we will get four terms: $|\mathcal{M}_V|^2$, $|\mathcal{M}_S|^2$, and the cross terms $\mathcal{M}_S^\dagger\mathcal{M}_V$ and $\mathcal{M}_V^\dagger\mathcal{M}_S$. Let's start with $|\mathcal{M}_S|^2$, summing over the final states of the lepton spins[2]:

$$\begin{aligned}\sum_{\text{spins}}|\mathcal{M}_S|^2 \equiv |\langle\mathcal{M}_S\rangle|^2 &= 2M(E'+M')|M_F|^2 2G_F^2\sum_{\text{spins}}\Big[v_s^\dagger(p_e)\big(C_S - C_S'\gamma_5\big)\\ &\qquad\times\big(\gamma^0 u_r(p_\nu)\big)\overline{u}_r(p_\nu)\big(C_S^* - C_S'^*\gamma_5\big)v_s(p_e)\Big]\\ &= 4M(E'+M')|M_F|^2G_F^2\sum_{\text{spins}}\Big[\big(v_s(p_e)\overline{v}_s(p_e)\big)\big(C_S + C_S'\gamma_5\big)\\ &\qquad\times\big(u_r(p_\nu)\overline{u}_r(p_\nu)\big)\big(C_S^* - C_S'^*\gamma_5\big)\Big]\\ &= 4M(E'+M')|M_F|^2G_F^2\ Tr\Big[(\not{p}_e - m_e)(C_S + C_S'\gamma_5)\\ &\qquad\times(\not{p}_\nu + m_\nu)(C_S^* - C_S'^*\gamma_5)\Big]\end{aligned} \tag{26}$$

[2]Since this is a $0^+ \longrightarrow 0^+$ transition, there are no spins to average over in the initial state.

Now recall some of the useful trace theorems:

$$Tr(\mathbf{AB}) = Tr(\mathbf{BA}) \tag{27a}$$
$$Tr(\text{odd number of } \gamma-\text{matrices}) = 0 \tag{27b}$$
$$Tr(\gamma_5) = Tr(\gamma_5 \not{a}) = Tr(\gamma_5 \not{a} \not{b}) = 0 \tag{27c}$$
$$Tr(\mathbb{I}) = 4 \tag{27d}$$
$$Tr(\not{a} \not{b}) = 4a \cdot b \tag{27e}$$
$$Tr(\not{a}\gamma^\mu \not{b}\gamma^\nu) = 4\big(a^\mu b^\nu + a^\nu b^\mu - a \cdot b\, g^{\mu\nu}\big) \tag{27f}$$

Using these, we can finish evaluating $|\langle \mathcal{M}_S \rangle|^2$:

$$\begin{aligned} |\langle \mathcal{M}_S \rangle|^2 &= 4M(E' + M')|M_F|^2 G_F^2 \; Tr\Big[\not{p}_e\big(C_S \not{p}_\nu C_S^* - C_S' \gamma_5 \not{p}_\nu C_S'^* \gamma_5\big) \\ &\qquad - m_e m_\nu \big(C_S C_S^* - C_S' \gamma_5 C_S'^* \gamma_5\big)\Big] \\ &= 4M(E' + M')|M_F|^2 G_F^2 \; Tr\Big[\not{p}_e \not{p}_\nu \big(|C_S|^2 + |C_S'|^2\big) \\ &\qquad - m_e m_\nu \big(|C_S|^2 - |C_S'|^2\big)\Big] \end{aligned} \tag{28}$$

$$\begin{aligned} |\langle \mathcal{M}_S \rangle|^2 = 16M(E' + M')E_e E_\nu |M_F|^2 G_F^2 \; &\Bigg[\big(|C_S|^2 + |C_S'|^2\big)\Big(1 - \frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu}\Big) \\ &- \big(|C_S|^2 - |C_S'|^2\big)\frac{m_e m_\nu}{E_e E_\nu}\Bigg] \end{aligned} \tag{29}$$

Now the vector case proceeds in the same manner:

$$\begin{aligned} |\langle \mathcal{M}_V \rangle|^2 &= 4M(E' + M')|M_F|^2 G_F^2 \sum_{\text{spins}} \Big[v_s^\dagger(p_e)(\gamma^0\gamma^0)\big(C_V + C_V'\gamma_5\big)\gamma^0 \\ &\qquad \times \big(\gamma^0 u_r(p_\nu)\big)\overline{u}_r(p_\nu)\gamma^0\big(C_V^* + C_V'^*\gamma_5\big)v_s(p_e)\Big] \\ &= 4M(E' + M')|M_F|^2 G_F^2 \sum_{\text{spins}} \Big[\big(v_s(p_e)\overline{v}_s(p_e)\big)\gamma^0\big(C_V + C_V'\gamma_5\big) \\ &\qquad \times \big(u_r(p_\nu)\overline{u}_r(p_\nu)\big)\gamma^0\big(C_V^* + C_V'^*\gamma_5\big)\Big] \\ &= 4M(E' + M')|M_F|^2 G_F^2 \; Tr\Big[(\not{p}_e - m_e)\gamma^0(C_V + C_V'\gamma_5) \\ &\qquad \times (\not{p}_\nu + m_\nu)\gamma^0(C_V^* + C_V'^*\gamma_5)\Big] \\ &= 4M(E' + M')|M_F|^2 G_F^2 \; Tr\Big[\not{p}_e\gamma^0\big(C_V \not{p}_\nu \gamma^0 C_V^* + C_V'\gamma_5 \not{p}_\nu \gamma^0 C_V'^* \gamma_5\big) \\ &\qquad - m_e\gamma^0\big(C_V m_\nu \gamma^0 C_V^* + C_V'\gamma_5 m_\nu \gamma^0 C_V'^*\gamma_5\big)\Big] \end{aligned}$$

$$\begin{aligned} |\langle \mathcal{M}_V \rangle|^2 = 4M(E' + M')|M_F|^2 G_F^2 &\Big[\big(|C_V|^2 + |C_V'|^2\big) Tr(\not{p}_e\gamma^0 \not{p}_\nu\gamma^0) \\ &- \big(|C_V|^2 - |C_V'|^2\big)m_e m_\nu Tr(\mathbb{I})\Big] \end{aligned}$$

$$\begin{aligned} |\langle \mathcal{M}_V \rangle|^2 = 16M(E' + M')E_e E_\nu |M_F|^2 G_F^2 &\Bigg[\big(|C_V|^2 + |C_V'|^2\big)\Big(1 + \frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu}\Big) \\ &- \big(|C_V|^2 - |C_V'|^2\big)\frac{m_e m_\nu}{E_e E_\nu}\Bigg] \end{aligned} \tag{30}$$

In the last line, we use the special case of Eq. (27f) where $\mu = \nu = 0$ to get $Tr(\not{p}_e\gamma^0 \not{p}_\nu\gamma^0) = E_e E_\nu + \vec{p}_e\vec{p}_\nu$.

Finally, we do the cross terms:

$$
\begin{aligned}
\langle\mathcal{M}_{VS}^2\rangle &\equiv \langle\mathcal{M}_V^\dagger\mathcal{M}_S\rangle + \langle\mathcal{M}_S^\dagger\mathcal{M}_V\rangle \\
&= 4M(E'+M')|M_F|^2G_F^2\sum_{\text{spins}}\Bigg[\Big[v_s^\dagger(p_e)\big(C_V+C_V'\gamma_5\big)\gamma^0 \\
&\qquad\times\big(\gamma^0u_r(p_\nu)\big)\overline{u}_r(p_\nu)\big(C_S^*-C_S'^*\gamma_5\big)v_s(p_e)\Big] \\
&\qquad+\Big[v_s^\dagger(p_e)\big(C_S-C_S'\gamma_5\big)\gamma^0u_r(p_\nu)\overline{u}_r(p_\nu)\gamma^0\big(C_V^*+C_V'^*\gamma_5\big)v_s(p_e)\Big]\Bigg] \\
&= 4M(E'+M')|M_F|^2G_F^2\ Tr\Big[(\not{p}_e-m_e)\gamma^0(C_V+C_V'\gamma_5)(\not{p}_\nu+m_\nu)\big(C_S^*-C_S'^*\gamma_5\big) \\
&\qquad+(\not{p}_e-m_e)(C_S+C_S'\gamma_5)(\not{p}_\nu+m_\nu)\gamma^0\big(C_V^*+C_V'^*\gamma_5\big)\Big] \\
&= 4M(E'+M')|M_F|^2G_F^2\ Tr\Big[\not{p}_e\gamma^0\big(C_Vm_\nu C_S^*-C_V'\gamma_5m_\nu C_S'^*\gamma_5\big) \\
&\qquad-m_e\gamma^0\big(C_V\not{p}_\nu C_S^*-C_V'\gamma_5\not{p}_\nu C_S'^*\gamma_5\big) \\
&\qquad+\not{p}_e(C_Sm_\nu\gamma^0C_V^*+C_S'\gamma_5m_\nu\gamma^0C_V'^*\gamma_5) \\
&\qquad-m_e\big(C_S\not{p}_\nu\gamma^0C_V^*+C_S'\gamma_5\not{p}_\nu\gamma^0C_V'^*\gamma_5\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\langle\mathcal{M}_{VS}^2\rangle = 4M(E'+M')|M_F|^2G_F^2\ Tr\Big[&\not{p}_e\gamma^0m_\nu\big(C_VC_S^*-C_V'C_S'^*+C_SC_V^*-C_S'C_V'^*\big) \\
&-\not{p}_\nu\gamma^0m_e\big(C_VC_S^*+C_V'C_S'^*+C_SC_V^*+C_S'C_V'^*\big)\Big] \qquad (31)
\end{aligned}
$$

Now note that

$$
\begin{aligned}
\not{p}\gamma^0 &= \left[\begin{pmatrix}E & 0\\ 0 & -E\end{pmatrix}-\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{p}\\ -\vec{\sigma}\cdot\vec{p} & 0\end{pmatrix}\right]\begin{pmatrix}\mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix} \\
&= \begin{pmatrix}E & \vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & E\end{pmatrix}, \qquad (32)
\end{aligned}
$$

$$\text{so}\qquad Tr(\not{p}\gamma^0) = 4E. \qquad (33)$$

Also note that only the real components of Eq. (31) survive (*i.e.* $C_SC_V^*+C_S^*C_V = 2\Re e(C_SC_V^*) = 2\Re e(C_S^*C_V)$, and similarly for the primed ones), so:

$$
\begin{aligned}
\langle\mathcal{M}_{VS}^2\rangle = 16M(E'+M')|M_F|^2G_F^2\Big[&2\Re e\big(C_SC_V^*-C_S'C_V'^*\big)m_\nu E_e \\
&-2\Re e\big(C_SC_V^*+C_S'C_V'^*\big)m_eE_\nu\Big]. \qquad (34)
\end{aligned}
$$

Finally, we put them all together to get the total matrix element:

$$
\begin{aligned}
|\langle\mathcal{M}_{fi}\rangle|^2 = \frac{1}{2}16M(E'+M')E_eE_\nu G_F^2\Bigg[& \\
&|M_F|^2\Big(|C_S|^2+|C_S'|^2+|C_V|^2+|C_V'|^2\Big)\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu} \\
&+2|M_F|^2\Re e\Big(C_SC_V^*-C_S'C_V'^*\Big)\frac{m_\nu}{E_\nu} \\
&-2|M_F|^2\Re e\Big(C_SC_V^*+C_S'C_V'^*\Big)\frac{m_e}{E_e} \\
&-|M_F|^2\Big(|C_S|^2-|C_S'|^2+|C_V|^2-|C_V'|^2\Big)\frac{m_em_\nu}{E_eE_\nu}\Bigg]. \qquad (35)
\end{aligned}
$$

This can be re-written as:

$$|\langle\mathcal{M}_{fi}\rangle|^2 = 8M(E'+M')E_eE_\nu G_F^2\ \xi\Bigg[1 + a\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu} + b_e\frac{m_e}{E_e} + b_\nu\frac{m_\nu}{E_\nu} + b_{e\nu}\frac{m_em_\nu}{E_eE_\nu}\Bigg], \tag{36}$$

$$\text{with :} \qquad \xi = \ |C_S|^2 + |C'_S|^2 + |C_V|^2 + |C'_V|^2 \tag{37a}$$

$$a\xi = \ |M_F|^2\Big(- |C_S|^2 - |C'_S|^2 + |C_V|^2 + |C'_V|^2\Big) \tag{37b}$$

$$b_e\xi = \ 2|M_F|^2\Re e\Big(C_SC_V^* + C'_SC'^*_V\Big) \tag{37c}$$

$$b_\nu\xi = -2|M_F|^2\Re e\Big(C_SC_V^* - C'_SC'^*_V\Big) \tag{37d}$$

$$b_{e\nu}\xi = \ |M_F|^2\Big(|C_S|^2 - |C'_S|^2 + |C_V|^2 - |C'_V|^2\Big). \tag{37e}$$

3 The Decay Rate

The decay rate is derived from Fermi's Golden Rule:

$$\Gamma = \int |\langle\mathcal{M}_{fi}\rangle|^2\frac{1}{2E_\mathrm{X}}\left[\frac{d^3\vec{k}}{(2\pi)^32E_\mathrm{Y}}\frac{d^3\vec{p}_\nu}{(2\pi)^32E_\nu}\frac{d^3\vec{p}_e}{(2\pi)^32E_e}\right](2\pi)^4\delta^4(p-k-p_e-p_\nu), \tag{38}$$

The matrix element, $\mathcal{M}_{fi}$, was derived in the previous section. Had we used the hadron current, Eq. (18), we would have ended up with a function, $F(Q^2, p\cdot p_\nu)$, which is given by Nieto [3]. For pure Fermi decays (all $G_i = 0$) and assuming no second class currents ($F_3 = 0$), his expression greatly simplifies to:

$$\begin{aligned} F(Q^2, p\cdot p_\nu) = f_1^2\Big[&2p\cdot p_\nu(M'^2 - M^2 + m_e^2 - 2p\cdot p_\nu) \\ &+ 2Q^2p\cdot p_\nu - \tfrac{1}{2}(Q^2+m_e^2)(Q^2+M'^2-M^2) + MM'(Q^2+m_e^2)\Big] \\ - f_2^2\Big[&\frac{Q^2+M^2+M'^2}{2M^2} + \frac{M'}{M}\Big]\Big[8(p\cdot p_\nu)^2 \\ &- 4p\cdot p_\nu(M'^2 - M^2 + m_e^2 + Q^2) + \tfrac{1}{2}(Q^2+m_e^2)\Big] \\ + f_1f_2\Big[&8(p\cdot p_\nu)^2(1+\tfrac{M'}{M}) - 2p\cdot p_\nu\big[(1+\tfrac{M'}{M})(2Q^2+m_e^2+2M'^2 \\ &- 2M^2) + 2m_e^2\big] - (2MM' - 2M^2 - m_e^2)(Q^2+m_e^2)\Big]. \end{aligned} \tag{39}$$

(believe it or not, this *is* simplified!)

So in what follows, we will use the simpler hadron current, and use the matrix element as given by Eq. (36).

3.1 Mandelstam Variables and Maximum Energies

As Ian shows in his report [4], it is useful to know the maximum energies of the decay products, and that we can obtain them from the Mandelstam variables, s, t and u. First, let us look at s:

$$s = -(p-p_\nu)^2 = -(k+p_e)^2 \tag{40}$$

$$M^2 - 2ME_\nu + m_\nu^2 = M'^2 + 2E'E_e - 2\vec{k}\cdot\vec{p}_e + m_e^2$$

$$E_\nu = \frac{1}{2M}\left(M^2 + m_\nu^2 - M'^2 - m_e^2 - 2E'E_e + 2\vec{k}\cdot\vec{p}_e\right) \tag{41}$$

The maximum neutrino energy, comparable to Ian's [4] Eq. (9), is obtained by setting $\vec{k} = \vec{p}_e = 0$, $E' = M'$ and $E_e = m_e$:

$$E_\nu^\mathrm{max} = \frac{M^2 + m_\nu^2 - (M'+m_e)^2}{2M}. \tag{42}$$

The maximum recoil energy can be derived from the t variable, which is just the negative square of the momentum transfer:

$$t = -(p-k)^2 = -(p_e + p_\nu)^2 \tag{43}$$

$$M^2 - 2ME' + M'^2 = m_e^2 + m_\nu^2 + 2E_eE_\nu - 2\vec{p}_e \cdot \vec{p}_\nu$$

$$E' = \frac{1}{2M}\left(M^2 + M'^2 - m_e^2 - m_\nu^2 - 2E_eE_\nu + 2\vec{p}_e \cdot \vec{p}_\nu\right). \tag{44}$$

This has a maximum of (setting $\vec{p}_e = \vec{p}_\nu = 0$, $E_e = m_e$ and $E_\nu = m_\nu$):

$$(E')^{\max} = \frac{M^2 + M'^2 - (m_e + m_\nu)^2}{2M}. \tag{45}$$

Finally, we get the β's energy from the u variable:

$$u = -(p - p_e)^2 = -(k + p_\nu)^2 \tag{46}$$

$$M^2 - 2ME_e + m_e^2 = M'^2 + 2E'E_\nu - 2\vec{k} \cdot \vec{p}_\nu + m_\nu^2$$

$$E_e = \frac{1}{2M}\left(M^2 + m_e^2 - M'^2 - m_\nu^2 - 2E'E_\nu + 2\vec{k} \cdot \vec{p}_\nu\right), \tag{47}$$

which has a maximum of (setting $\vec{k} = \vec{p}_\nu = 0$, $E' = M'$ and $E_\nu = m_\nu$):

$$E_e^{\max} = \frac{M^2 + m_e^2 - (M' + m_\nu)^2}{2M} \tag{48}$$

and let us define, for a massless neutrino:

$$A_\circ \equiv E_e^{\max}(m_\nu = 0) = \frac{M^2 + m_e^2 - M'^2}{2M} \tag{49}$$

Of course, as with the maximum β energy, all of these expressions agree with Ian's report [4] if we set $m_\nu = 0$. Note, however, how the symmetry of Eqs. (42), (45) and (48) is made more apparent when the neutrino is not considered massless.

3.2 Integrations

Integrating Eq. (38) over $d^3\vec{k}$ using the δ^3-function we get:

$$\Gamma = \int \frac{G_F^2 |\langle \mathcal{M}_{fi} \rangle|^2}{(2\pi)^5 4ME'} \frac{d^3\vec{p}_e}{2E_e} \frac{d^3\vec{p}_\nu}{2E_\nu} \delta(M - E' - E_e - E_\nu). \tag{50}$$

We will now attack the integration over the unobserved neutrino energy using the δ-function. We start by defining a function to be its argument:

$$f(E_\nu) = M - E'(E_\nu) - E_e - E_\nu, \tag{51}$$

so that we can use the following property of the δ-function to do the integration over dE_ν:

$$\int \delta\left[f(E_\nu)\right] = \int \frac{1}{|f'(E_\nu^n)|} \, \delta(E_\nu - E_\nu^n), \tag{52}$$

where E_ν^n are the roots of the equation $f(E_\nu) = 0$.

As made obvious in Eq. (51), the recoil energy is a function of the neutrino's energy, and therefore it's momentum. So, let's put it in terms of p_ν:

$$E' = \sqrt{k^2 + M'^2}, \qquad \vec{k} = \vec{p}_e + \vec{p}_\nu \tag{53}$$

$$= \sqrt{(\vec{p}_e + \vec{p}_\nu)^2 + M'^2}$$

$$= [p_e^2 + p_\nu^2 + 2p_ep_\nu \cos\theta_{e\nu} + M'^2]^{1/2}, \tag{54}$$

so that the derivative of E' with respect to E_ν is

$$\frac{d}{dE_\nu}E' = \frac{1}{\cancel{2}E'}(\cancel{2}p_\nu + \cancel{2}p_e\cos\theta_{e\nu})\frac{dp_\nu}{dE_\nu}$$

$$= \frac{(p_\nu + p_e\cos\theta_{e\nu})E_\nu}{p_\nu E'}, \tag{55}$$

where we used $\frac{dp_\nu}{dE_\nu} = \frac{E_\nu}{p_\nu}$. The derivative of $f(E_\nu)$ is therefore:

$$f'(E_\nu) = -\left(1 + \frac{E_\nu(p_\nu + p_e\cos\theta_{e\nu})}{p_\nu E'}\right), \tag{56}$$

so that

$$\begin{aligned}\frac{1}{|f'(E_\nu)|} &= \frac{p_\nu E'}{p_\nu E' + E_\nu p_\nu + E_\nu p_e \cos\theta_{e\nu}} \\ &= \frac{E'}{E' + E_\nu + \frac{E_\nu}{p_\nu} p_e\cos\theta_{e\nu}} \\ &= \frac{M - E_e - E_\nu}{M - E_e + \frac{E_\nu}{p_\nu} p_e\cos\theta_{e\nu}} \\ &\approx 1 - \frac{1}{M}\left(E_\nu + \tfrac{E_\nu}{p_\nu} p_e \cos\theta_{e\nu}\right).\end{aligned} \tag{57}$$

Now, we need to find the roots of Eq. (51), E_ν^n. It is at this point that allowing a massive neutrino complicates matters beyond what I have considered. The problem comes from the $\vec{p}_\nu$ term in Eq. (44). I think Ian ends up saying that if the neutrino isn't massless, one needs to do a derivation akin to that of Koefed-Hanson [5], which is pretty messy ...

So, taking $E_\nu = p_\nu$ and dropping terms of order $1/M^2$ and $1/M'^2$, we substitute Eq. (44) into Eq. (51) to get:

$$\begin{aligned} f(E_\nu^n) = 0 &= M - E_e - E_\nu^n - \tfrac{M^2 + M'^2 - m_e^2 - 2E_eE_\nu^n + 2p_e\cos\theta_{e\nu}E_\nu^n}{2M} \\ E_\nu^n\left[1 + \tfrac{1}{2M}\left(2p_e\cos\theta_{e\nu} - 2E_e\right)\right] &= M - E_e - \tfrac{1}{2M}\left(M^2 + M'^2 - m_e^2\right) \\ E_\nu^n \tfrac{1}{M}\left(M - E_e + p_e\cos\theta_{e\nu}\right) &= \tfrac{1}{2M}\left(\not{2}M^2 - \not{M}^2 - M'^2 + m_e^2\right) - E_e \\ E_\nu^n &= \frac{A_\circ - E_e}{\frac{1}{M}(M - E_e + p_e\cos\theta_{e\nu})} \\ E_\nu^n &= \frac{M(A_\circ - E_e)}{M - E_e + p_e\cos\theta_{e\nu}} \\ &\approx (A_\circ - E_e)\left[1 + \frac{1}{M}(E_e - p_e\cos\theta_{e\nu})\right]. \end{aligned} \tag{58}$$

At this point, I'd just like to point out the comparison of this expression (which is the same as Ian Towner's Eq. 20 [4]) to that of Eq. 2.12 in Ortiz's thesis [6]:

$$(E_\nu^n)_{\text{Ortiz}} = \frac{M^2 - (M' + m_e)^2 + m_e^2 - 2ME_e}{2(M - E_e + p_e\cos\theta_{e\nu})} \tag{59}$$

$$\begin{aligned} &= \frac{M^2 - M'^2 + m_e^2 - 2ME_e}{2(M - E_e + p_e\cos\theta_{e\nu})} - \frac{2m_eM' + m_e^2}{2(M - E_e + p_e\cos\theta_{e\nu})} \\ &= E_\nu^n - \frac{m_e(2M' + m_e)}{2(M - E_e + p_e\cos\theta_{e\nu})}. \end{aligned} \tag{60}$$

Since I derived Ian's expression from scratch and understand each step, I believe it is correct, and not Ortiz's which is decidedly different. The fact that they differ essentially by the electron's mass cannot be coincidental ....

Now we finish evaluating the decay rate ... from Eq. (38),

$$\Gamma = \int dE_e d\Omega_e d\Omega_\nu \, \frac{G_F^2}{(2\pi)^5}\left[1 - \frac{E_\nu^n + p_e\cos\theta_{e\nu}}{M}\right] p_\nu^n E_\nu^n \; p_e E_e \; \frac{|\langle \mathcal{M}_{fi}\rangle|^2}{16ME'E_eE_\nu}. \tag{61}$$

Let us evaluate the terms involving E_ν^n and simplify:

$$
\begin{aligned}
\left[1-\tfrac{E_\nu^n+p_e\cos\theta_{e\nu}}{M}\right](E_\nu^n)^2 &= \left[1-\frac{1}{M}(A_\circ-E_e)\big(1+\tfrac{E_e}{M}-\tfrac{p_e\cos\theta_{e\nu}}{M}\big)\right] \\
&\quad\times\big(A_\circ-E_e\big)^2\left[1+\frac{1}{M}(E_e-p_e\cos\theta_{e\nu})\right]^2 \\
&\approx (A_\circ-E_e)^2\left[1-\frac{A_\circ-E_e+p_e\cos\theta_{e\nu}}{M}\right]\left[1+\frac{2}{M}(E_e-p_e\cos\theta_{e\nu})\right] \\
&\approx (A_\circ-E_e)^2\left[1-\frac{A_\circ-E_e+p_e\cos\theta_{e\nu}}{M}+\frac{2}{M}(E_e-p_e\cos\theta_{e\nu})\right] \\
&= (A_\circ-E_e)^2\left[1+\frac{3E_e-A_\circ-3p_e\cos\theta_{e\nu}}{M}\right],
\end{aligned} \tag{62}
$$

so that Eq. (61) becomes:

$$
\Gamma=\int dE_e d\Omega_e d\Omega_\nu\,\frac{G_F^2}{(2\pi)^5}p_eE_e(A_\circ-E_e)^2\left[1+\frac{3E_e-A_\circ-3p_e\cos\theta_{e\nu}}{M}\right]\frac{|\langle\mathcal{M}_{fi}\rangle|^2}{16ME'E_eE_\nu}. \tag{63}
$$

3.3 The Differential Decay Rate

We've done everything now. All that is left is to put our expression for $|\langle\mathcal{M}_{fi}\rangle|^2$, Eq. (36), into the decay rate, Eq. (63). Including the Fermi function, which is briefly described in the next section, the differential decay rate for a pure Fermi β^+ transition including recoil-order corrections (but not massive neutrinos or radiative corrections) is:

$$
\begin{aligned}
\frac{d^5\Gamma}{dE_ed\Omega_ed\Omega_\nu} &= \frac{G_F^2}{(2\pi)^5}p_eE_e(A_\circ-E_e)^2\left[1+\frac{3E_e-A_\circ-3p_e\cos\theta_{e\nu}}{M}\right] \\
&\quad\times\frac{1}{16ME'E_eE_\nu}8M(E'+M')E_eE_\nu\xi\left[1+a\frac{\vec{p}_e\cdot\vec{p}_\nu}{E_eE_\nu}+b_e\frac{m_e}{E_e}\right]F(E_e,Z',R)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^5\Gamma}{dE_ed\Omega_ed\Omega_\nu} &= \frac{G_F^2}{(2\pi)^5}p_eE_e(A_\circ-E_e)^2\left[1-\frac{A_\circ-3(E_e-p_e\cos\theta_{e\nu})}{M}\right]\left[\frac{1}{2}\Big(1+\tfrac{M'}{E'}\Big)\right] \\
&\quad\times\xi\left[1+a\frac{p_e}{E_e}\cos\theta_{e\nu}+b_e\frac{m_e}{E_e}\right]F(E_e,Z',R).
\end{aligned} \tag{64}
$$

3.4 Massive ν's

The decay rate, Eq. (64), is only valid for $m_\nu=0$, but we need *some* sort of expression that includes it for M. Trinczek's analysis [7]. Consider Eq. (4.20) of reference [8]:

$$
\begin{aligned}
\frac{d\Gamma}{dE_e} &\propto F(E_e,Z',R)p_eE_e\sum_{i=1}^{N}|U_{ei}|^2\Big[E_e^{\max}(0)-E_e\Big]^2 \\
&\quad\times\left[1-\frac{m_{\nu_i}^2}{\big[E_e^{\max}(0)-E_e\big]^2}\right]^{\frac{1}{2}}\Theta\big(E_e^{\max}(m_{\nu_i})-E_e\big)
\end{aligned} \tag{65}
$$

where $\Theta(x)=1$ if $x>0$ (otherwise zero) is the usual step function, $E_e^{\max}(m_\nu)$ is given by Eq. (47), and U_{ei} is the mixing strength of an electron neutrino with that of any of N massive ν's; this is defined by relating the weak eigenstates, $\nu_j,(j=1,2,\ldots,N)$ to the physical ν_i $(i=e,\mu,\tau,\ldots)$:

$$
\nu_i=\sum_j U_{ij}\nu_j \tag{66}
$$

In the case of an electron-neutrino mixing with one type of heavy neutrino ($N=2$), we get a 2×2 matrix which can be characterized by one parameter; can define a *mixing angle*, θ:

$$
\begin{pmatrix}\nu_e\\ \nu_H\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}, \tag{67}
$$

or, wee can define $U_{eH} \equiv \sin\theta$ so that:

$$\begin{pmatrix} \nu_e \\ \nu_H \end{pmatrix} = \begin{pmatrix} \sqrt{1-|U_{eH}|^2} & U_{eH} \\ -U_{eH} & \sqrt{1-|U_{eH}|^2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \tag{68}$$

Thus we get the decay rate:

$$\frac{d\Gamma}{dE_e} \propto \begin{cases} F(E_e, Z', R) p_e E_e \big[A_\circ - E_e\big]^2 & \text{if } E_e \geq E_e^{\max}(m_{\nu_H}) \quad (69a) \\ F(E_e, Z', R) p_e E_e \big[A_\circ - E_e\big]^2 \left[\big(1-|U_{eH}|^2\big) + |U_{eH}|^2 \sqrt{1 - \frac{m_{\nu_H}^2}{(A_\circ - E_e)^2}}\right] & \text{if } E_e < E_e^{\max}(m_{\nu_H}) \quad (69b) \end{cases}$$

where we have substituted $A_\circ \equiv E_e^{\max}(m_{\nu_H} = 0)$.

Going back to the start of §3.2 (integrating the decay rate), the δ-function in Eq. (50) is simple to solve if we neglect recoil-order corrections; it is simply given by dropping the $1/M$ term in the approximation of Eq. (58):

$$E_\nu = A_\circ - E_e. \tag{70}$$

Now we define the magnitude of the neutrino's momentum as:

$$|\vec{p}_\nu| = \sqrt{E_\nu^2 - m_\nu^2}, \tag{71}$$

which is true whether $m_\nu = 0$ or not. We can then use just one (*i.e.* simple!) expression for the differential decay rate, namely (dropping the mixing strength for the moment):

$$\frac{d^5\Gamma(m_\nu)}{dE_e d\Omega_e d\Omega_\nu} \equiv \frac{G_F^2}{(2\pi)^5}\, p_e E_e\, |\vec{p}_\nu| E_\nu\, F(E_e, Z', R)\, \xi \left[1 + a\frac{\vec{p}_e \cdot \vec{p}_\nu}{E_e E_\nu} + b_e \frac{m_e}{E_e} + b_\nu \frac{m_\nu}{E_\nu} + b_{e\nu}\frac{m_e m_\nu}{E_e E_\nu}\right], \tag{72}$$

(the parameters b_ν and $b_{e\nu}$ are included for generality but should be set to zero, and experiments limit $0 \leq |b_e| < 0.007$). The only sublety is in how we interpret it, and when we include a neutrino mass. Breaking up the problem into two energy regimes (*i.e.* whether mixing is allowed energetically or not):

1. If $E_e \geq E_e^{\max}(m_{\nu_H})$, then no mixing can occur and the decay rate is:

$$\frac{d^5\Gamma_{\text{massive}}}{dE_e d\Omega_e d\Omega_\nu} = \frac{d^5\Gamma(m_\nu = 0)}{dE_e d\Omega_e d\Omega_\nu} \tag{73}$$

   and since $m_\nu = 0$ in this case, $|\vec{p}_\nu| = E_\nu$. Also in this case, one gets the familiar $a\frac{v_e}{c}\cos\theta_{e\nu}$ term in the angular distribution.

2. If $E_e < E_e^{\max}(m_{\nu_H})$, then we have a different probability function. On the basis of Eq. (69b), the decay rate with no mixing is only affected by the global normalization $(1 - |U_{eH}|^2)$ factor, and we add in the decay rate for events where mixing has occured:

$$\frac{d^5\Gamma_{\text{massive}}}{dE_e d\Omega_e d\Omega_\nu} = \underbrace{\frac{d^5\Gamma(m_\nu = 0)}{dE_e d\Omega_e d\Omega_\nu}}_{\text{massless } \nu} \big(1 + |U_{eH}|^2\big) + \underbrace{\frac{d^5\Gamma(m_\nu = m_{\nu_H})}{dE_e d\Omega_e d\Omega_\nu}}_{\text{massive } \nu} |U_{eH}|^2. \tag{74}$$

   This says, then, that event generation of $m_\nu = 0$ is calculated exactly the same as when $E_e \geq E_e^{\max}(m_{\nu_H})$, but the total decay rate in this case contains additional strength to generate massive ν's; the expression is the same as for $m_\nu = 0$, but $|\vec{p}_\nu| \neq E_\nu$ and the rate is supressed by the square of the mixing strength, $|U_{eH}|^2$.

As far as my MC is concerned, the decay rate was implemented in the following way: first check if decay allows mixing energetically. If it does not, then just generate a $m_\nu = 0$ event. If mixing can occur, then calculate the decay rate for $m_\nu = 0$ and compare it to a randomly generated number between 0 and the maximum decay rate (which includes both terms in Eq. (74); calculated upon initialization). If the random number is less than the decay rate, generate a $m_\nu = 0$ event; otherwise, check if the random number is less than the *sum* of the decay rates of the $m_\nu = 0$ and $m_\nu = m_{\nu_H}$ cases. If it is, then generate an event with $m_\nu = m_{\nu_H}$ (otherwise start all over).

4 The Fermi Function

The "traditional" function includes relativistic effects, but treats the nucleus as infinitely massive, evaluates the integrals only at the nuclear surface, R, and does not include screening effects. With $\gamma = \sqrt{1 - \alpha^2 Z'^2}$, $\alpha = e^2/4\pi$ and $\eta = \pm E_e/p_e$, it is given by:

$$F(E_e, Z', R) = (2\gamma + 1)(2p_e R)^{2(\gamma-1)} \; e^{\pi\eta} \; \frac{|\Gamma(\gamma + i\eta)|^2}{|\Gamma(1 + 2\gamma)|^2}, \tag{75}$$

where Z' refers to the the daughter nucleus and η is negative for β^+ decays. The nuclear radius can be given in two ways, the simplest being the well known:

$$R = 1.2A^{1/3} \text{ fm}. \tag{76}$$

However, if the root-mean-square (RMS) charge radius, R_{RMS}, is known, it is better to assume the charge is homogeneously distributed within the spherical nuclear volume so that:

$$R^2 = \frac{5}{3} R^2_{\text{RMS}}. \tag{77}$$

It seems a waste of time to retype the lengthy results of Wilkinson's parameterizations, so the reader is referred to [9]. Let me just outline a couple of points one should know when reading this paper:

- He works in units where $m_e = \hbar = c = 1$, *i.e.*

$$W = E_e/m_e, \tag{78a}$$

$$W_\circ = A_\circ/m_e, \tag{78b}$$

$$\text{and } p = \sqrt{W^2 - 1}. \tag{78c}$$

- His quote that "$R = 2.5896 \times 10^{-3} \times R$ [fm]" is simply the conversion:

$$R \text{ [unitless]} = \frac{m_e}{\hbar c} R \text{ [fm]}. \tag{79}$$

- He does not include screening effects, but this is can be included on the basis of the simple Eq. 6 in [10].
- I have not looked closely at [11]; I really should one day . . .

Figure 2 shows a plot of the Fermi functions with differing amounts of corrections applied.

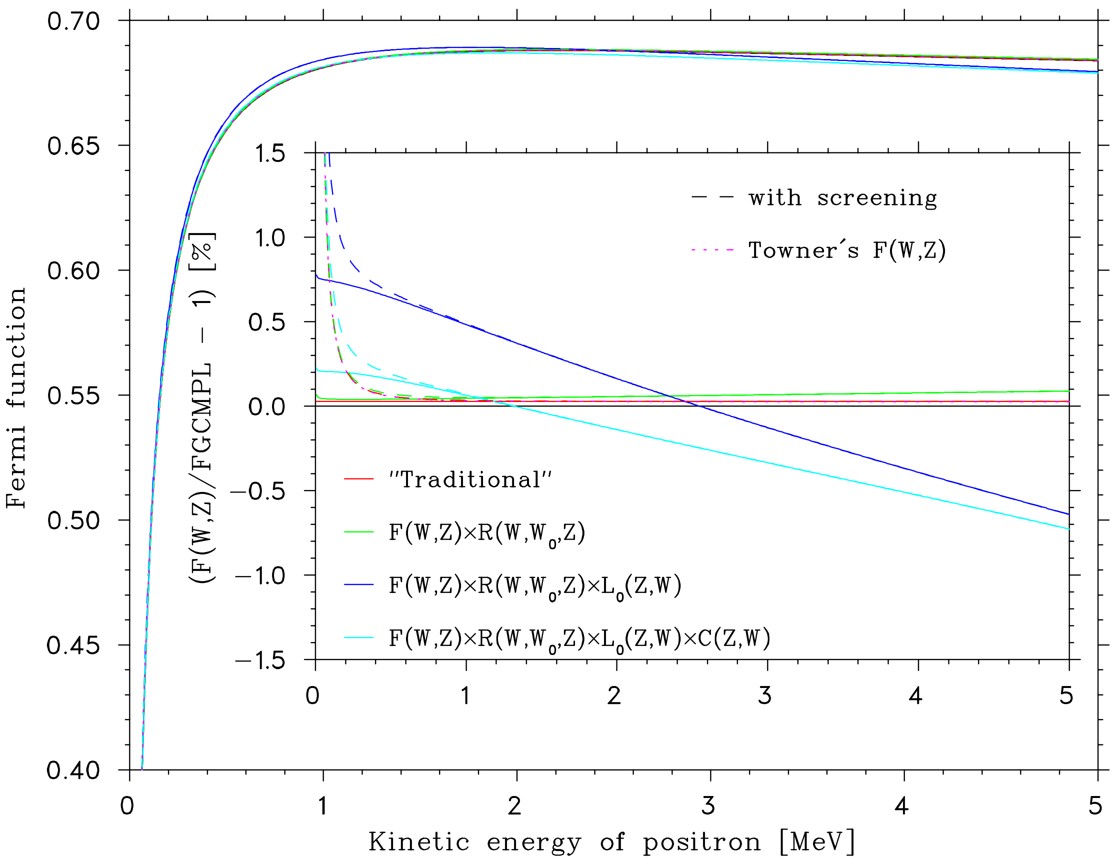


Figure 2: Comparison of corrections to the traditional Fermi function for β^+ decay.

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