Implementing recoil-order corrections

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Abstract

This report summarizes work done on understanding and implmenting recoil-order corrections, specifically for ^{37}K. The result is coded in the program `recoil-order.c` [1], which has been checked against the results quoted by Naviliat-Cuncic and Severijns in Ref. [2]. Some relatively minor corrections to both that paper and Holstein's original treatise [3] on the subject are discussed, and results for ^{37}K are presented.

The main impetus for working on this was to be able to properly update Naviliat-Cuncic and Severijns' paper [2] given our improved ^{37}K lifetime measurement [4]. Of course, recoil-order corrections are not significant at this stage, but an eye is being kept forward to the future when TRINAT will make a precise enough measurement of the correlation parameters that they do become important. Additionally, this let's us quantifiably determine what impact on $|V_{\rm ud}|$ we may have, and how sensitive we are to second-class currents.

1 The Recoil-Order Corrections

The lengthy and mathematically intense review on the subject of recoil-order corrections to allowed β decay by Holstein [3] is of course our starting point. I have never tried – and am likely never to try – to derive his decay rates from first principles, and so assume his expressions for $F(E, u, v, s)$ are all correct. We begin by noting the minor errors in Holstein's paper and then determine the correspondance between Holstein's parameters with the correlation parameters of Jackson, Treiman and Wyld (JTW) [7, 8]. This section finishes with plots of these correlation parameters as a function of β energy.

1.1 Corrections to the spin-dependent functions

In addition to the erratum published two years after his original paper, there are a few other errors that have been discovered in Holstein's original paper and which have not been published. As I noted previously [5], his simplified expressions for the spin-dependent functions, Eqs (B8) of Ref. [3], are not all correct. We *do* assume, however, that the expressions in terms of the Racah coefficient, $W(j_1j_2l_2l_1;j_3l_3)$ *are* correct; hopefully they are.

As described in Ref. [5], the relevant corrections needed here are:

- The expressions for $\lambda_{u,v}$ all need to be divided by $-5\sqrt{6}$. It is only non-zero if $u+v \geqslant 2$. For the specific case if $I = I'$, for example, $\lambda_{I,I} = -\frac{1}{5}\sqrt{(2I-1)(2I+3)/6}$.
- The functions for ϵ_{uv} are all rather wrong ... there's no simple fix like with λ_{uv}.

Table 1: Summary of form-factors used by Holstein and the symmetry of the current to which they belong.

Symbol	Current	2^{nd} class?	Name	associated paramters
g_V	vector	N	the (usual) vector form factor	a, b, e, f, g
g_A	axial	N	the (usual) axial-vector form factor	c, d, h, j_i
g_M	vector	N	weak magnetism	b
g_P	axial	N	induced pseudoscalar	h
g_S	vector	Y	induced scalar	e
g_{II}	axial	Y	induced tensor	d

The corrected expressions are:

$$\epsilon_{u,v} = \frac{-2}{\sqrt{70}} \begin{cases} \frac{1}{(2u-1)} \left[\frac{(2u-3)(2u-2)(2u+2)}{2u+3} \right]^{1/2} & u = v{+}1;\ u \geqslant 2 \\ \left[\frac{3(2u-2)(2u+4)}{2(2u-1)(2u+3)} \right]^{1/2} & u = v; \quad u \geqslant \frac{3}{2} \\ \frac{1}{(2u+3)} \left[\frac{(2u) \quad (2u+4)(2u+5)}{2u-1} \right]^{1/2} & u = v{-}1;\ u \geqslant 1 \end{cases}$$

1.2 Understanding the decay rate

The heart of the matter for TRINAT is the decay rate, Eq. (51), in Holstein's Rev. Mod. Phys. paper [3]. In some ways, it is not as general as the seminal papers by JTW [7, 8] in that Holstein assumes a (V, A) interaction; JTW's papers allow for any current consistent with Lorentz invariance and so in addition to vector and axial vector, include scalar and tensor interactions (pseudoscalar is automatically supressed and cannot contribute to β decay). Holstein's paper is, however, more complete than JTW's in that he has not made the assumption that the recoiling nucleus is infinitely heavy; he has made the so-called recoil-order corrections, terms that go like the β decay energy over the nuclear mass. Other terms that appear like scalar, pseudoscalar and tensor interactions are *induced* when including recoil terms.

First, it should be mentioned and will be used later that Holstein uses the representation of the Dirac matrices as defined by Bjorken and Drell [6] except for the sign of γ_5. This effectively means that Holstein has a $V + A$ interaction instead of $V - A$; said another way, in Holstein's convention, all axial currents have the sign opposite to someone who uses Bjorken and Drell's convention including their γ_5. Near the beginning of §V of Holstein's paper, he described what his various form factors are. The separation between vector and axial-vector form factors is most manifest in his Eq. (66). Table 1 summarizes their classification and description. I therefore propose that if $c \approx g_A M_{GT}$ changes sign due to choice of the γ matrices, then so also must d, h and $j_i (i = 1, 3)$ since these too are also associated with axial currents. I have *not* confirmed this with an explicit calculation, but it seems clear that it is true (anyone is invited to to do the calculation if they have any reservations about this assertion).

We now turn to Eq. (51) of Holstein's review. It gives the decay rate for allowed β decay when the parent is polarized and both momenta of the leptons are observed. It is a very long equation in terms of many spectral functions, $f_i(E) = F_i(E, u, v, 0)$, where E is the β's *total* energy and u, v are the initial/final spins of the nucleus. The $F_i(E, u, v, s)$ functions are given in his appendix B6 in terms of the parameters listed in Table 1. We readily and easily identify $f_1(E)$ as corresponding to JTW's ξ once recoil-order corrections are included; in the limit that $E \to 0$ or $M \to \infty$, and that the weak interaction is purely $V - A$:

$$f_1(E) \xrightarrow[M\to\infty]{E\to 0} |a_1|^2 + |c_1|^2 = g_A^2|M_{GT}|^2 + g_V^2|M_F|^2 \equiv \xi. \tag{1}$$

The only other unpolarized/non-aligned term in the equation is the following one and, since it goes like $\boldsymbol{p} \cdot \hat{k}$ (where $\hat{k} = \boldsymbol{p}_\nu / E_\nu$ is the direction of the neutrino momentum), leads to the identification $f_2(E) = \xi\, a_{\beta\nu}$, or

$$a_{\beta\nu}(E) = \frac{f_2(E)}{f_1(E)} \tag{2}$$

Naïvely, one might then look at this equation and conclude that $f_4(E)$, which multiplies a term that goes like $\frac{\langle \boldsymbol{I} \rangle}{I} \cdot \frac{\boldsymbol{p}}{E}$, corresponds to $\xi\, A_\beta$. However, as Holstein explicity points out in the following section, "knowledge of the neutrino momentum involves the difficult job of detecting nuclear recoil, so that most experiments involve an average over neutrino momenta." In his Eq. (52) which is this neutrino-averaged decay rate, the term multiplying $\frac{\langle \boldsymbol{I} \rangle}{I} \cdot \frac{\boldsymbol{p}}{E}$ has become* $F_1(E)$. It is not difficult to do the integration for ones self and confirm that $F_1(E) = f_4(E) + \frac{1}{3} f_7(E)$ (terms linear in $\hat{k}$ average to zero). Thus A_β including recoil-order corrections is

$$A_\beta(E) = \frac{H_1(E)}{H_0(E)} = \frac{f_4(E) + \frac{1}{3} f_7(E)}{f_1(E)}. \tag{3}$$

What do we do about the neutrino asymmetry parameter, B_ν, though? Naviliat-Cuncic and Severijns [2] *wrongly* assume it is $h_6(E)/f_1(E)$, where $h_6(E)$ is given in another paper by Holstein, Ref. [9] which, neverthless, is equal to $f_6(E)$ from his later Rev. of Mod. Phys. paper. In neither case did Holstein (nor anyone else as far as I'm aware) average over electron momenta; so $h_6(E)$ aka $f_6(E)$ as used by Naviliat-Cuncic and Severijns is *not* the correct expression. We have to integrate Holstein's decay rate over $d\Omega_e$ ourselves and determine the factor multiplying $\frac{M_I}{I} \hat{n} \cdot \hat{k}$. It's not a big deal, though; this time all terms linear in $\boldsymbol{p}$ average to zero. Either doing it explicity or even just based on inspection, one can easily tell that Naviliat-Cuncic and Severijns should add $f_5(E)/3f_1(E)$ to their Eq. (15), *i.e.*

$$B_\nu(E) = \frac{f_6(E) + \frac{1}{3} f_5(E)}{f_1(E)}. \tag{4}$$

Finally, we find the term in Holstein's Eq. (51) corresponding to JTW's alignment term by finding the one that is the traceless symmetric 2^{nd}-rank tensor constructed out of the β and neutrino momenta, *i.e.* $T^{(2)}(\hat{n}) : [\boldsymbol{p}, \hat{k}]$. The only term this corresponds to is the one multiplied by $f_{12}(E)$. Note that since both $\boldsymbol{p}_e$ and $\boldsymbol{p}_\nu$ appear in the alignment term of JTW, that there is *no* averaging over any lepton momentum in this case; this is the only term. The correspondence of Holstein's $f_{12}(E)$ with JTW's alignment parameter c_{align} is a little more work than one might imagine due to how they differ in their definitions

*Which is not to be confused with $F_1(E, u, v, s)$! $F_1(E) = H_0(E, u, v, 0)$ is a combination of $F_i(E, u, v, 0)$'s as given by his Eqs. (B7).

of statisical population functions and 2$^{\text{nd}}$-rank tensors; however, comparing Holstein's expression with JTW's and equating, we see that

$$\Lambda^{(2)}T^{(2)}(\hat{n})\!:\left[\frac{\boldsymbol{p}}{E},\hat{k}\right]\cdot f_{12}(E) = \xi c_{\text{align}}\left(\frac{\boldsymbol{p}_e\cdot\boldsymbol{p}_\nu}{3E_eE_\nu} - \frac{(\boldsymbol{p}_e\cdot\boldsymbol{i})(\boldsymbol{p}_\nu\cdot\boldsymbol{i})}{E_eE_\nu}\right)\left(\frac{I(I+1)-3\langle I^2\rangle}{I(2I-1)}\right) \quad (5)$$

which leads to

$$c_{\text{align}}(E) = -\frac{f_{12}(E)}{f_1(E)}\left(\frac{2I-1}{I+1}\right). \quad (6)$$

1.3 Correlation parameters of $\overrightarrow{^{37}\text{K}}$

The above equations have been coded into the program `recoil-order.c` with experimental inputs and the shell-model predictions of I. Towner for ^{37}K as listed in Table 2. Most of the input parameters listed come from Holstein's paper, Ref. [3]; others relating to calculating $|V_{ud}|$ are from Severijns *et al.*'s Ref. [10]. Note that using the dipole moments of Table 2 instead of the compilation [11] used by Naviliat-Cuncic and Severijns, we arrive at a weak magnetism of $b = -45.03(4)$ instead of $-44.99(24)$. It seems unlikley that Naviliat-Cuncic and Severijns included the induced quadrupole, however we do and with the electric quadrupole moments listed in Table 2, we find $g = -1.36(17)\times 10^5$.

Figures 1–4 show the results of the calculation, including varying different parameter values, namely the ratio of matrix elements, $\rho = c_1/a_1$, possible second-class currents (SCCs) via d (I do not yet know how to differentiate between d_{I} and d_{II}) and the limit on the precision of the predictions due to the measured electric dipole and quadrupole moments. As one can see, we are still heavily dominated by uncertainty in ρ, with theoretical uncertainties not contributing much to the asymmetries. To be sensitive to SCC, we many need to improve both the branching ratio and lifetime measurements of ^{37}K, however measuring the β asymmetry as a function of energy to 0.2% or so (the best case) seems daunting .... It is unfortunate that A_β and B_ν show the least variation with β energy; measuring c_{align} as a funcion of energy is all but impossible, however the energy-dependence of $a_{\beta\nu}$ is both large and experimentally accessible; we should reconsider how difficult it will be to prepare a random spin-population and measure this correlation. Note, however, that unlike A_β, SCC do not change the slope of the energy dependence, and so in that case determing ρ and measuring the value absolutely are more important.

2 Turning Things Around: Determining $|V_{ud}|$

As pointed out explicitly by Naviliat-Cuncic and Severijns [2], one may use $T = 1/2$ mirror transitions for which a measurement of a correlation parameter has been done to determine a value of $|V_{ud}|$ which may complement the value deduced from $0^+ \longrightarrow 0^+$ decays. The program `recoil-order.c` also calculates these values, and finds (when the same inputs are used) almost perfect agreement with Naviliat-Cuncic and Severijns.

One finds, with the inputs of Table 2 and the new lifetime for ^{37}K, $f_V t = 4565(7)$ s or

$$\mathcal{F}t = 4594(8)\text{ s}. \quad (7)$$

This can be compared to the old lifetime (with an old value for b) which led to $\mathcal{F}t = 4562(28)$ s. The change in the central value of the lifetime leads to the shift in the $\mathcal{F}t$,

Table 2: Inputs used for the calculation of the correlation parameters for ^{37}K. When applying this to Holstein's expressions, each of c_i, d, h, j_1, j_2 and j_3 are passed with the opposite sign, as discussed earlier.

$\frac{M_1+M_2}{2} = 36.97007611(12)$ amu	$E_\circ = 5.63646(23)$ MeV	$\langle E \rangle = 3.35$ MeV
$f_V = 3623.9(7)$	$f_A/f_V = 1.00456(91)$	$I = 3/2$
$t_{1/2}{}^\dagger = 1.2335(69)$ s	branch $= 97.99(14)\%$	$P_{EC} = 0.080(2)\%$
$\delta_C = 0.73(6)\%$	$\delta_{\mathrm{NS}} = -0.06(2)\%$	$\delta'_R = 1.431(39)\%$
$a_1 = 1$	$a_2 = 0$	$b = A\sqrt{\frac{I+1}{I}} M_F \left(\frac{\mu-\mu'}{T_3-T'_3}\right)$
$c_1 = 0.5794(20)$ (from $\mathcal{F}t$)	$\mu(^{37}\mathrm{K}) = 0.20321(6)\mu_N$	$\mu(^{37}\mathrm{Ar}) = 1.146(1)\mu_N$
$c_2 = 1.764$	$d = 0 \pm 0.4Ac$	$e = 0$ (by CVC)
$f = 0$ (CVC); -3.394 (Towner)	$g = -M_F\sqrt{\frac{(I+1)(2I+3)}{I(2I-1)}}\frac{2M^2}{3\hbar^2c^2}(Q-Q')$	
$h = -4.10 \times 10^4$	$Q(^{37}\mathrm{K}) = 10.6(4)e\,\mathrm{fm}^2$	$Q(^{37}\mathrm{Ar}) = 7.6(9)e\,\mathrm{fm}^2$
$j_1 = -1.97 \times 10^5$	$j_2 = 0.0121$	$j_3 = 3.99 \times 10^5$

†Preliminary new value [4] is an order-of-magnitude more precise than the previous world average of 1.2248(73) s (excludes G. Ball's unpublished result).

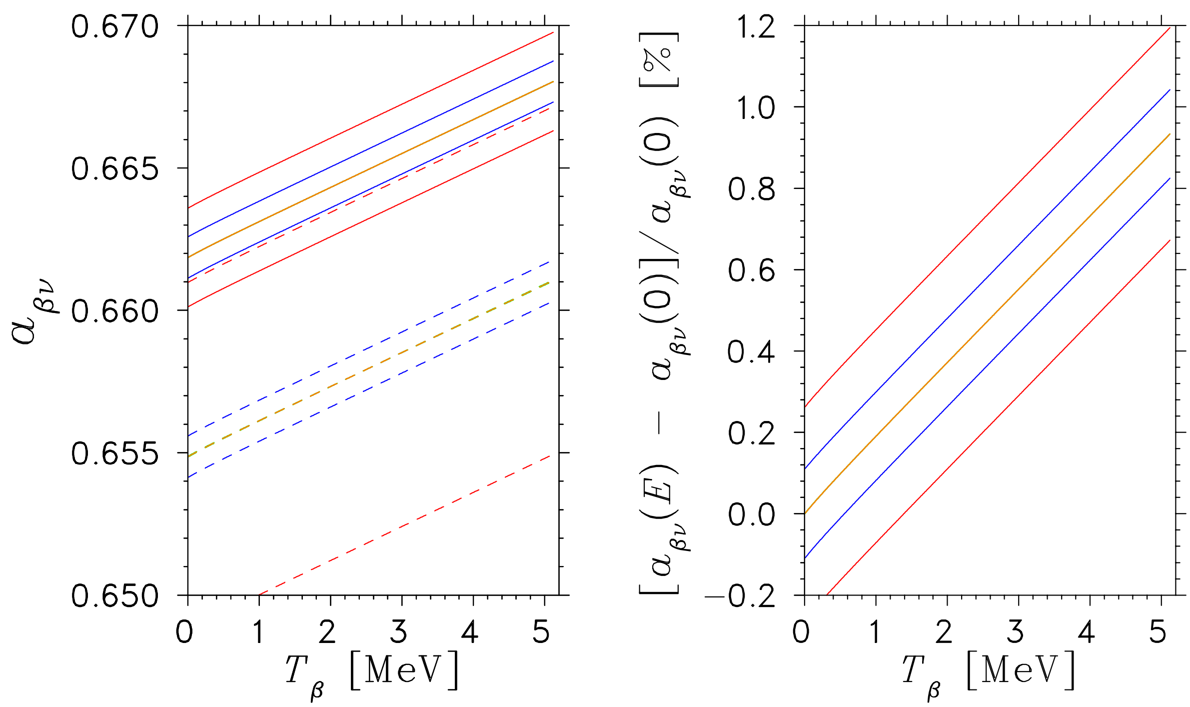

Figure 1: The value of the $\beta - \nu$ correlation parameter with the old (dashed) and new (solid) value for the lifetime [left], and the deviation from $a_{\beta\nu}$ at $E = m_e$ [right]. The green curves are the values spanned by the uncertainty in weak magnetism, b; the orange represents the uncertainty in the induced quadrupole, g; the green is the change one would see if second-class currents were at their limit, $d/Ac = \pm 0.4$; and the red curves cover the range due to the uncertainty in $\rho = c_1/a_1$ as determined from the $\mathcal{F}t$ value.

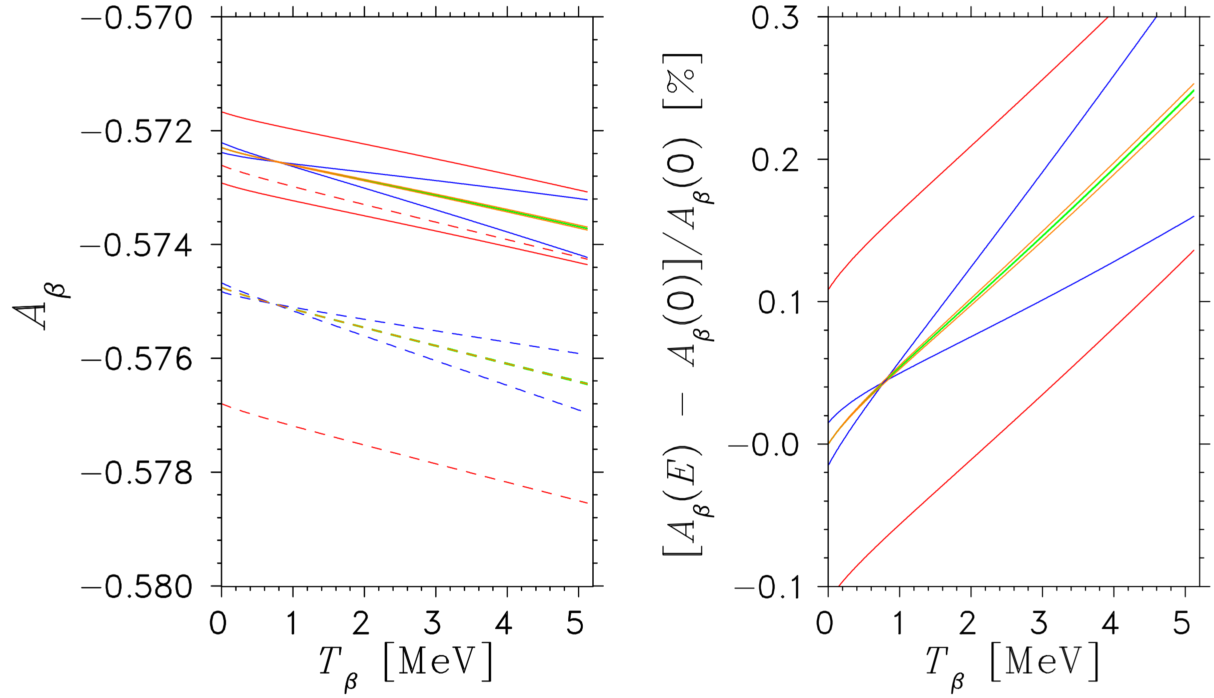

Figure 2: Plots similar to Fig. 1 except of the β asymmetry parameter.

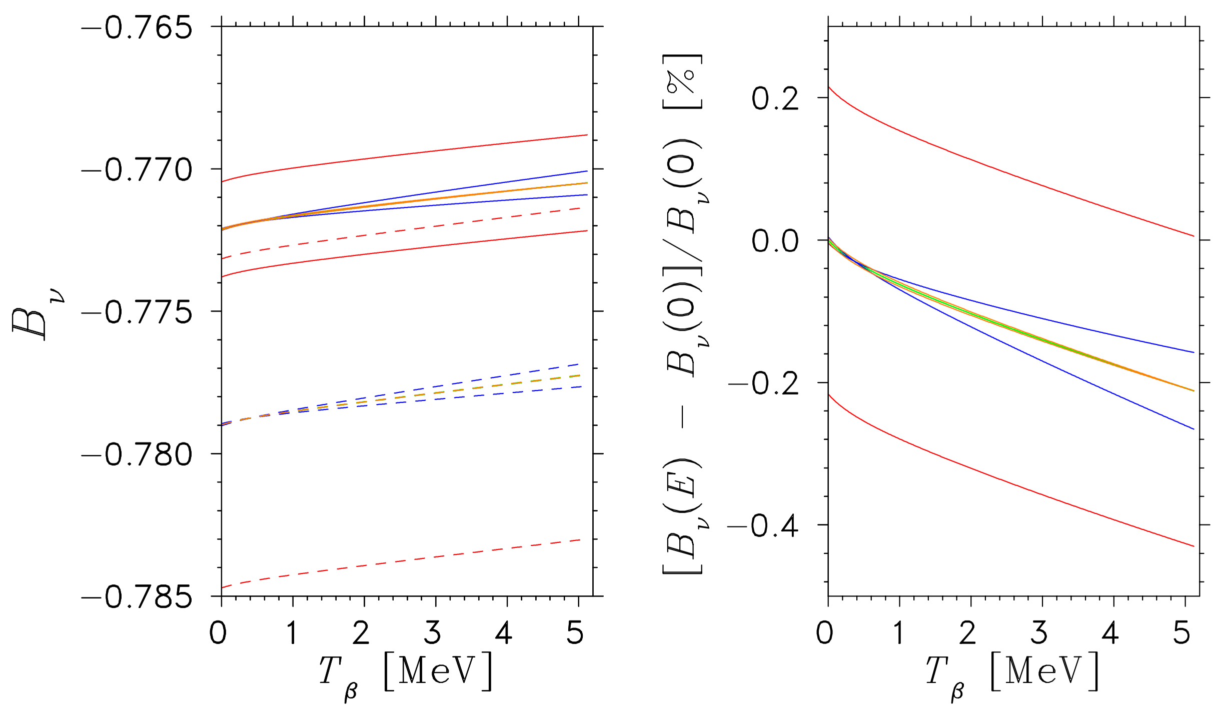

Figure 3: Plots similar to Fig. 1 except of the ν asymmetry parameter.

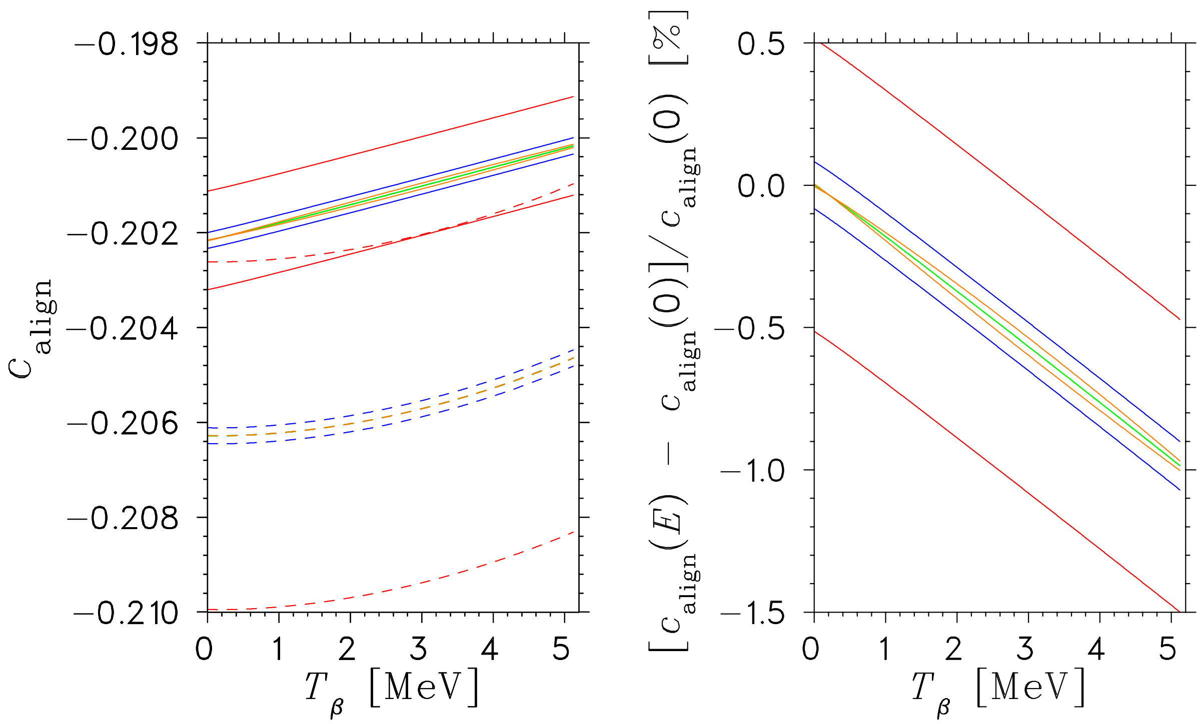

Figure 4: Plots similar to Fig. 1 except of the alignment parameter.

and the order-of-magnitude improvement in the lifetime leads to a 3.5× reduction in its uncertainty. Given this and using the value from averaging over $0^+ \longrightarrow 0^+$ decays, $\mathcal{F}t = 3071.81(79)(27)$ s, we determine ρ to have the value listed in Table 2 (and which was used earlier in calculating the correlation coefficients). Instead, however, we use the value of the neutrino asymmetry as measured by TRINAT [12]: $B_\nu = -0.755(20)(13)$ which is actually $-0.7547(227)$ to get

$$\rho = 0.5603^{+0.0269}_{-0.0253} = 0.561(26), \tag{8}$$

which leads to

$$\mathcal{F}t_0 = 6043(135)\ \text{s}, \tag{9}$$

which is only slightly better precision compared to what is quoted by Naviliat-Cuncic and Severijns prior to the lifetime re-measurement; the central value again changes because of the new lifetime, but in fact the reduced uncertainty in $\mathcal{F}t_0$ is only due to the more detailed value for B_ν I have access to that they didn't. To improve this $\mathcal{F}t_0$ value rests entirely on measuring B_ν (or any other correlation parameter) to better precision.

Finally, the $\mathcal{F}t_0$ value in turn leads to a new value for $|V_{ud}|$ for ^{37}K:

$$|V_{ud}| = 0.9823(110) \tag{10}$$

which, compared to the older value 0.9857(119), approaches and is better aligned with the average value of other mirror transitions: 0.9710(22) (^{19}Ne), 0.9696(35) (^{21}Ne), 0.9445(643) (^{29}P) and 0.9756(39) (^{35}Ar). These results are shown graphically in Fig. 5 and for comparison $V_{ud} = 0.97425(22)$ deduced from $0^+ \longrightarrow 0^+$ decays is also shown (blue). Due to the large uncertainty in ^{37}K compared to most of the others, it does not change the average.

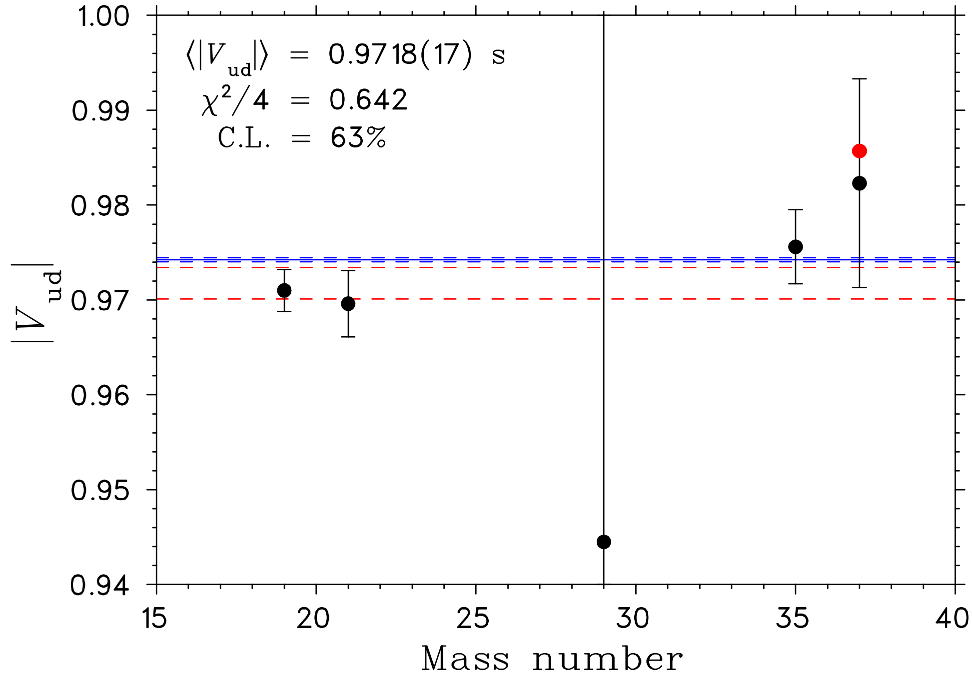

Figure 5: Values of $|V_{ud}|$ from $T = 1/2$ mirror transitions, adapted from Naviliat-Cuncic and Severijns [2]. The red point is the previous value as quoted in Ref. [2] and has the same uncertainty as the new value and so is not shown for clarity. The green dashed lines show the uncertainty in the average value from mirror transitions, while the result from $0^+ \longrightarrow 0^+$ decays is shown in blue.

A Expressions for the relevant spectral functions

We assume all of Holstein's parameters are real in what follows. We consider $a_1 a_2$ and $c_1 c_2$ terms, but drop the very small a_2^2 and c_2^2 terms in the radiative corrections.

$$
\begin{aligned}
f_1(E) + \Delta f_1(E) = c_1 \Bigg\{ & 2c_2 \left[\left(\frac{20E(E_\circ - E) + m_e^2(11 - \frac{2E_\circ}{E})}{9M^2} \right) \right. \\
& \left. \pm \frac{8\alpha Z}{3\pi} \left(\tfrac{1}{3} E_\circ X - 4E(\tfrac{4}{3}X + Y) - \frac{m_e^2}{E}(X + 2Y) \right) \right] \\
& - d \left(\frac{2E_\circ - \frac{m_e^2}{E}}{3M} \right) \pm b \left(\frac{2(2E - E_\circ) - \frac{m_e^2}{E}}{3M} \right) + h \left(\frac{m_e^2(E_\circ - E)}{6M^2 E} \right) \Bigg\} \\
+ c_1^2 \Bigg\{ & 1 - \frac{2(E_\circ - 5E - \frac{m_e^2}{E})}{3M} \pm \frac{8\alpha Z}{3\pi} \left(\frac{1}{3} E_\circ X - 4E(\tfrac{4}{3}X + Y) - \frac{m_e^2}{E}(X + 2Y) \right) \Bigg\} \\
+ a_1^2 \Bigg[& 1 + \frac{2E}{M} \mp \frac{8\alpha Z}{3\pi} \left(4E(X + Y) + E_\circ X + \frac{m_e^2}{E}(X + 2Y) \right) \Bigg] \\
& + a_1 \left[2a_2 \left(\frac{4E(E_\circ - E) + m_e^2(1 + \frac{2E_\circ}{E})}{3M^2} \mp \frac{8\alpha Z}{3\pi} \left\{ 4E(X + Y) + E_\circ X + \frac{m_e^2}{E}(X + 2Y) \right\} \right) \right. \\
& \left. + e \left(\frac{m_e^2}{ME} \right) \right] \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
f_2(E) + \Delta f_2(E) = c_1 \Bigg\{ & d\left(\frac{2E_\circ}{3M}\right) \pm b\left(\frac{2(E_\circ - 2E)}{3M}\right) \\
& +2c_2\left[\pm\frac{8\alpha Z}{3\pi}\left(\tfrac{4}{3}E(2X+Y) - E_\circ X\right) - \frac{8E(E_\circ - E) + m_e^2}{3M^2}\right]\Bigg\} \\
& + c_1^2\left\{-\frac{1}{3} + \frac{2(E_\circ - 2E)}{3M} \pm \frac{8\alpha Z}{3\pi}\left(\tfrac{4}{3}E(2X+Y) - E_\circ X\right)\right\} \\
& + a_1^2\left[1 \mp \frac{8\alpha Z}{3\pi}\left(\tfrac{4}{3}E(2X+Y) - E_\circ X\right)\right] \\
& + 2a_1a_2\left[\frac{m_e^2}{M^2} \mp \frac{8\alpha Z}{3\pi}\left(\tfrac{4}{3}E(2X+Y) - E_\circ X\right)\right]
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
f_4(E) + \Delta f_4(E) + \frac{1}{3}f_7(E) = c_1 \Bigg\{ & \sqrt{\frac{I}{I+1}}\left[a_1\left(2 - \frac{4(E_\circ - 7E)}{3M}\right) + 2a_2\frac{4E(E_\circ - E) + 3m_e^2}{3M^2}\right] \\
& \pm \frac{\gamma_{I,I}}{I+1}\left[d\frac{2E_\circ + E}{3M} \pm b\frac{2E_\circ - 5E}{3M} - 2c_2\frac{8E(E_\circ - E) + m_e^2}{3M^2}\right] \\
& - \frac{\lambda_{I,I}}{I+1}\Bigg[f\frac{5E}{M} + g\sqrt{\frac{3}{2}}\left(\frac{E_\circ(E_\circ - 11E) + 5E^2 + 6m_e^2}{6M^2}\right) \\
& \qquad \pm 3j_2\left(\frac{E_\circ(E_\circ - E) - 4E^2 + 2m_e^2}{4M^2}\right)\Bigg] \\
& + \frac{8\alpha Z\, E(5X+4Y)}{3\pi}\left(2c_2\frac{\gamma_{I,I}}{I+1} \mp \sqrt{\frac{I}{I+1}}2(a_1 + a_2)\right)\Bigg\} \\
& \mp c_1^2\left\{\frac{\gamma_{I,I}}{I+1}\left(1 - \frac{E_\circ - 4E}{3M} + \frac{8\alpha Z\, E(5X+4Y)}{3\pi}\right)\right\} \\
& - 2a_1\sqrt{\frac{I}{I+1}}\Bigg[(d \pm b)\left(\frac{E_\circ - E}{M}\right) + h\frac{m_e^2}{4M^2} \\
& \qquad + c_2\left(\frac{8E(E_\circ - E) + 3m_e^2}{3M^2} \pm \frac{16\alpha Z\, E(5X+4Y)}{3\pi}\sqrt{\frac{I}{I+1}}\right)\Bigg]
\end{aligned} \tag{13}
$$

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